The Study of Porous Medium Equations with Other Initial Conditions by Using Transform Methods

Maturada Grappananon, Pilasluck Sornkaew

Divison of Mathematics, Faculty of Science, Maejo University, Thailand

Abstract: In this paper, we study a comparison of transform methods to solve porous medium equations with other initial conditions. The nonlinear term can be handled by adomian decomposition method. The results tell us that the good method is more efficient and easier to handle porous medium equations with other initial conditions.

Keywords: Laplace Transform, Elzaki Transform, Sumudu Transform, Adomian Decomposition Method, Porous Medium Equations

1. Introduction

In recent years, the partial differential equations have been used in many of the physical phenomena and various fields of engineering and science. The porous medium equations are the nonlinear heat equation to describe various physical phenomena, the form given by [1]

$$u_{t}(x,t) = \frac{\partial}{\partial x} \left((u(x,t))^{m} u_{x}(x,t) \right)$$
(1)

Where m is a rational number. There are number of physical applications where this model appears in a natural way to describe fluid flow, heat transfer or diffusion.

In 2013 Bhadane and Pradhan [2] proposed Elzaki transform homotopy perturbation method to solve porous medium equation when m=1 with initial condition as u(x,0) = x in Example 3.2 and they get the solution is u(x,t) = x+t

In this paper, we use Laplace transform, Elzaki transform and Sumudu transform to solve porous medium equation when m=1 and the nonlinear term in the equations are handled by Adomian decomposition method with other initial conditions. We compare between transform methods to found that method are good for solving this equation with other initial condition.

2. Basic Definitions

2.1. Adomain's Polynomials [6]

Definition Let E and F be two Banach spaces, K a scalar field, define $N: E \to F, u \to Nu$ a nonlinear operator, differentiable up to the n^{th} order at $\Phi(\lambda n)$ where Φ is a function of the scalar variable λ , taking its values in E, and defined by

$$\Phi(\lambda) = \sum_{i=0}^n \lambda^i u_i$$

We define the Adomian's polynomials by the formula

$$A_n\left\{N(u)\right\} = \frac{1}{n!} \frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^n \lambda^i u_i\right)\Big|_{\lambda=0}, n \in \square$$

2.2. Laplace Transform [3]

$$L[f(t)] = \int_{0}^{\infty} e^{-st} f(t) dt = F(s)$$

In general F(s) will exist for $s > \alpha$ where α is some constant. L is called the Laplace transform operator.

2.3. Elzaki Transform [4]

Consider functions in the set A defined by

$$A = \left\{ f(t) \mid \exists M, k_1, k_2 > 0 \mid f(t) < Me^{|t|/k_j}, t \in (-1)^j \times [0, \infty) \right\}$$

The Elzaki transform is defined by

$$E[f(t)] = u^{2} \int_{0}^{\infty} f(ut)e^{-t}dt = T(u), u \in (k_{1}, k_{2})$$

2.4. Sumudu Transform [5]

Over the set of functions

$$A = \left\{ f(t) \mid \exists M, \tau_1, \tau_2 > 0 \mid f(t) < Me^{|t|/\tau_j}, t \in (-1)^j \times [0, \infty) \right\}$$

The Sumudu transform is defined by

$$G(u) = S[f(t)] = \int_{0}^{\infty} f(ut)e^{-t}dt, u \in (-\tau_{1}, \tau_{2})$$

3. Applications

In this section show the effectiveness of the Laplace transform, Elzaki transform and Sumudu transform with Adomian polynomials.

Example 3.1 From equation (1) when m = 1 we get

$$u_t(x,t) = \frac{\partial}{\partial x} \left(u(x,t)u_x(x,t) \right)$$
⁽²⁾

with initial condition as u(x, 0) = x

Applying the Laplace transform

we have

$$u_{t}(x,t) = u(x,t)u_{xx}(x,t) + (u_{x}(x,t))^{2}$$
(3)

we apply the Laplace transform on both sides of (3), we have

$$L[u_t(x,t)] = L\left[u(x,t)u_{xx}(x,t) + \left(u_x(x,t)\right)^2\right]$$
(4)

This can be written as

$$sL[u(x,t)] - u(x,0) = L\left[u(x,t)u_{xx}(x,t) + (u_{x}(x,t))^{2}\right]$$
(5)

According to the initial condition, we can obtain

$$L[u(x,t)] = \frac{x}{s} + \frac{1}{s} L[u(x,t)u_{xx}(x,t) + (u_{x}(x,t))^{2}]$$
(6)

Taking inverse Laplace transform on both sides of (6) we get

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$$u(x,t) = x + L^{-1} \left[\frac{1}{s} L \left[u(x,t) u_{xx}(x,t) + \left(u_x(x,t) \right)^2 \right] \right]$$
(7)

Now we apply Adomian decomposition method

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$
(8)

And the nonlinear term can be decomposed as

$$N(u(x,t)) = \sum_{n=0}^{\infty} A_n(u)$$
⁽⁹⁾

According to (7)-(9) is equivalent to

$$\sum_{n=0}^{\infty} u_n(x,t) = x + L^{-1} \left[\frac{1}{s} L \left[\sum_{n=0}^{\infty} A_n(u) \right] \right]$$

where $A_n(u)$ are Adomian polynomials. The first few components of Adomian polynomials are given by

$$A_{0}(u) = u_{0}u_{0xx} + (u_{0x})^{2}$$

$$A_{1}(u) = (u_{1}u_{0xx} + u_{0}u_{1xx}) + 2u_{0x}u_{1x}$$
:

We get

$$u_{0}(x,t) = x$$

$$u_{1}(x,t) = L^{-1} \left[\frac{1}{s} L[A_{0}(u)] \right]$$

$$= L^{-1} \left[\frac{1}{s} L\left[u_{0} \frac{\partial^{2} u_{0}}{\partial x^{2}} + \left(\frac{\partial u_{0}}{\partial x} \right)^{2} \right] \right]$$

$$= t$$

$$u_{2}(x,t) = L^{-1} \left[\frac{1}{s} L[A_{1}(u)] \right]$$

$$= L^{-1} \left[\frac{1}{s} L[(u_{1}u_{0xx} + u_{0}u_{1xx}) + 2u_{0x}u_{1x}] \right] = 0$$

$$u_{3}(x,t) = 0$$

$$u_{4}(x,t) = 0$$

$$\vdots$$

$$u_{n}(x,t) = 0$$
Therefore, the solution of (2) is

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$
$$= x + t$$

<u>Applying the Elzaki transform</u> we apply the Elzaki transform on both sides of (3), we have

$$E\left[u_{t}(x,t)\right] = E\left[u(x,t)u_{xx}(x,t) + \left(u_{x}(x,t)\right)^{2}\right]$$

This can be written as

$$\frac{E[u(x,t)]}{u} - u(u(x,0)) = E\left[u(x,t)u_{xx}(x,t) + \left(u_{x}(x,t)\right)^{2}\right]$$

According to the initial condition, we can obtain

$$E[u(x,t)] = xu^{2} + uE[u(x,t)u_{xx}(x,t) + (u_{x}(x,t))^{2}]$$
(10)

Taking inverse Elzaki transform on both sides of (10) we get

$$u(x,t) = x + E^{-1} \left[u E \left[u(x,t) u_{xx}(x,t) + \left(u_{x}(x,t) \right)^{2} \right] \right]$$
(11)

Now we apply Adomian decomposition method

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$
(12)

And the nonlinear term can be decomposed as

$$N(u(x,t)) = \sum_{n=0}^{\infty} A_n(u)$$
(13)

According to (11)-(13) is equivalent to

$$\sum_{n=0}^{\infty} u_n(x,t) = x + E^{-1} \left[uE\left[\sum_{n=0}^{\infty} A_n(u)\right] \right]$$

where $A_n(u)$ are Adomian polynomials. We get

$$u_{0}(x,t) = x$$

$$u_{1}(x,t) = E^{-1} \left[uE[A_{0}(u)] \right]$$

$$= E^{-1} \left[uE[u_{0}u_{0xx} + (u_{0x})^{2}] \right]$$

$$= t$$

$$u_{2}(x,t) = E^{-1} \left[uE[A_{1}(u)] \right]$$

$$= E^{-1} \left[uE[(u_{1}u_{0xx} + u_{0}u_{1xx}) + 2u_{0x}u_{1x}] \right] = 0$$

$$u_{3}(x,t) = 0$$

$$u_{4}(x,t) = 0$$

$$\vdots$$

$$u_{n}(x,t) = 0$$
Therefore, the solution of (2) is

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$
$$= x+t$$

<u>Applying the Sumudu transform</u> we apply the Sumudu transform on both sides of (3), we have

$$S[u_t(x,t)] = S[u(x,t)u_{xx}(x,t) + (u_x(x,t))^2]$$

This can be written as

$$\frac{S[u(x,t)]}{u} - \frac{u(x,0)}{u} = S[u(x,t)u_{xx}(x,t) + (u_x(x,t))^2]$$

According to the initial condition, we can obtain

$$S[u(x,t)] = x + uS[u(x,t)u_{xx}(x,t) + (u_x(x,t))^2]$$
(14)

Taking inverse Sumudu transform on both sides of (14) we get

$$u(x,t) = x + S^{-1} \left[uS \left[u(x,t)u_{xx}(x,t) + \left(u_{x}(x,t)\right)^{2} \right] \right]$$
(15)

Now we apply Adomian decomposition method

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$
(16)

And the nonlinear term can be decomposed as

$$N(u(x,t)) = \sum_{n=0}^{\infty} A_n(u)$$
(17)

According to (15)-(17) is equivalent to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\sum_{n=0}^{\infty} u_n(x,t) = x + S^{-1} \left[uS\left[\sum_{n=0}^{\infty} A_n(u)\right] \right]$$

where $A_n(u)$ are Adomian polynomials. We get

$$u_{0}(x,t) = x$$

$$u_{1}(x,t) = S^{-1} \left[uS[A_{0}(u)] \right]$$

$$= S^{-1} \left[uS[u_{0}u_{0xx} + (u_{0x})^{2}] \right]$$

$$= t$$

$$u_{2}(x,t) = S^{-1} \left[uS[A_{1}(u)] \right]$$

$$= S^{-1} \left[uS[(u_{1}u_{0xx} + u_{0}u_{1xx}) + 2u_{0x}u_{1x}] \right] = 0$$

$$u_{3}(x,t) = 0$$

$$u_{4}(x,t) = 0$$

$$\vdots$$

$$u_{n}(x,t) = 0$$
Therefore, the collution of (2) is

Therefore, the solution of (2) is

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$
$$= x+t$$

Remark consider initial conditions u(x,0) = x+t or tthe solution is 1) Laplace : u(x,t) = x+2t or u(x,t) = t2) Elzaki : u(x,t) = x+2t or u(x,t) = t

3) Sumudu :
$$u(x,t) = x + 2t$$
 or $u(x,t) = t$, respectively.

Example 3.2 From equation (2) with initial condition as $u(x, 0) = e^x$ In the same way with example 3.1,

by the Laplace transform we get the solution is $u(x,t) = e^x + 2te^{2x} + 9t^2e^{3x} + ...,$ the Elzaki transform we get the solution is $u(x,t) = e^x + 2te^{2x} + 9t^2e^{3x} + ...$ and the Sumudu transform we get the solution is $u(x,t) = e^x + 2te^{2x} + 9t^2e^{3x} + ...$ **Remark** consider initial conditions $u(x,0) = e^{x+t}$ or e^t the solution is 1) Laplace : $u(x,t) = e^{x+t} - e^{2x} + e^{2x+2t} + 6e^{3x} - 9e^{3x+t} + 3e^{3x+3t} + ... \text{ or } u(x,t) = e^t$ 2) Elzaki : $u(x,t) = e^{x+t} - e^{2x} + e^{2x+2t} + 6e^{3x} - 9e^{3x+t} + 3e^{3x+3t} + ... \text{ or } u(x,t) = e^t$ 3) Sumudu : $u(x,t) = e^{x+t} - e^{2x} + e^{2x+2t} + 6e^{3x} - 9e^{3x+t} + 3e^{3x+3t} + ... \text{ or } u(x,t) = e^t$, respectively.

4. Conclusion

The main goal of this paper is to show a comparison of transform methods to solve porous medium equations with other initial conditions. We found that all method can solve that and to get the same solution. When we consider the initial condition is $u(x,0) = e^{x+t}$, we found that some term of equation is hard to solve by the general Elzaki and Sumudu . This term must be decomposed and that is easy to solve by Elzaki and Sumudu transform.

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